NOTE ON SYSTEMS OF DIFFERENTIAL EQUATIONS REGARDING MON APR 20 CLASS

If you find any errors or typos in this note, please alert me.

For people in the 11am lecture, TYPO:
At the very end, when I wrote down the initial value constraint, it should have (clearly) been

$$\vec{x}(0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

instead of

$$\vec{x}(t) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}.$$

(namely t was supposed to be 0)

And note, $\vec{x}(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$, so what $\vec{x}(0) = \begin{pmatrix} 1 \\ 6 \end{pmatrix}$ means is that $x_1(0) = 1$ and $x_2(0) = 6$.

For everyone:
I realized that in our discussion about what solutions to the linear system of differential equations are expressed as

$$\vec{x}(t) = e^{\lambda_1 t}\mathbf{u} + e^{\lambda_2 t}\mathbf{v}$$

where $\mathbf{u}$ can be *any* eigenvector for the eigenvalue λ_1 and $\mathbf{v}$ can be *any* eigenvector for the eigenvalue λ_2, we not only do not get $\vec{x}(t) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, but we also do not get the solutions of the form

$$\vec{x}(t) = e^{\lambda_1 t}\mathbf{u} \quad \text{or} \quad \vec{x}(t) = e^{\lambda_2 t}\mathbf{v},$$

which we know are solutions (that's the first thing we proved.)

These are the cases $a = 0$ and $b = 0$ from the general solution

$$\vec{x}(t) = ae^{\lambda_1 t}\mathbf{u} + be^{\lambda_2 t}\mathbf{v},$$

where now a, b can be *any* constants, and $\mathbf{u}$ and $\mathbf{v}$ in this last expression are particular choices of eigenvectors for λ_1 and λ_2, respectively.

So if instead of letting the constants a, b range through all real numbers, we let the eigenvectors $\mathbf{u}$ and $\mathbf{v}$ range through all possible eigenvectors for λ_1 and λ_2, then we would miss all the solutions that correspond to the cases $a = 0$ or $b = 0$ or both $a = b = 0$.

Here is a quick review of how we came up with the general solution (I want you to understand this, because you'll be using it over and over again when you solve systems of differential equations, and I want you to understand things you are doing.)

- we showed that for *any* eigenvalue λ and *any* eigenvector $\mathbf{u}$ for λ,
$$\vec{x}(t) = e^{\lambda t}\mathbf{v}$$
is a solution.
- we showed that if $\vec{y}(t)$ and $\vec{z}(t)$ are solutions, then also
$$a\vec{y}(t) + b\vec{z}(t)$$
is also a solution for *any* real numbers a and b.
- the two above facts combined say that if A has distinct real eigenvalues λ_1 and λ_2, then for *any* eigenvector $\mathbf{u}$ for λ_1 and for *any* eigenvector $\mathbf{v}$ for λ_2, and for *any* real numbers a, b,
$$\vec{x}(t) = ae^{\lambda_1 t}\mathbf{u} + be^{\lambda_2 t}\mathbf{v}$$
is a solution.
- now we noted that that's a bit overkill – we noted it's enough to just *pick one* eigenvector for λ_1 and *pick one* eigenvector for λ_2 because if a, b range through all real numbers, in particular they range through all nonzero real numbers, and then a times our choice of eigenvector will range though all eigenvectors for λ_1, and similarly for λ_2.

Thus the general solution is
$$\vec{x}(t) = ae^{\lambda_1 t}\mathbf{u} + be^{\lambda_2 t}\mathbf{v},$$
where $\mathbf{u}$ is a choice of eigenvector for λ_1 and $\mathbf{v}$ is a choice of eigenvector for λ_2, but a, b can be ANY real numbers.